Math 564: Advance Analysis 1
Lecture 1

Motivation for measure theory.

From probability. We can do probability on finite sets, such as n coinflips:

0 is with prob. 1/3
1 is — — — 2/3.

$$\underbrace{\boxed{0/1}\boxed{0/1}\boxed{0/1}\boxed{\,.\,}\boxed{\,.\,}\boxed{\,.\,}\cdots}_{n}$$

The prob of each word $w \in 2^n$ is $\left(\frac{1}{3}\right)^{\#\text{of } 0s} \cdot \left(\frac{2}{3}\right)^{\#\text{of } 1s}$.

We would like to take $n = \infty$, i.e. the space of infinite sequences of 0s and 1s and answers questions like: what's the probability that a random word $w \in 2^{\mathbb{N}}$ doesn't have 00111 as a subword? Turns out the answer is 0, but to answer it we need a notion of probability on $2^{\mathbb{N}}$ that "extends" the notion of $1/3$ / $2/3$ probability on 2^n for each n.

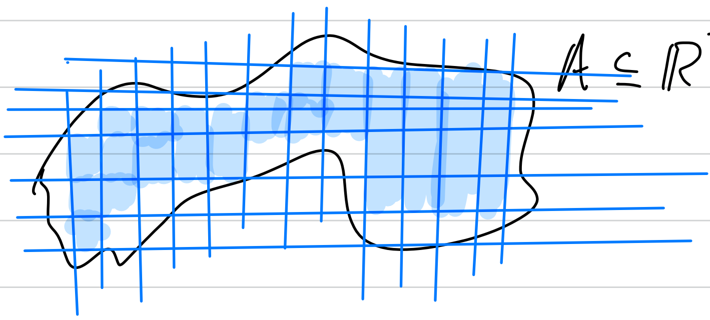
From geometry. After all, we want to have a notion of

- length for subsets of $\mathbb{R}$
- area for subsets of $\mathbb{R}^2$
- volume for subsets of $\mathbb{R}^3$

$\vdots$

Say for $\mathbb{R}^2$, we can compute the area of rectangles $I \times J$, where $I, J \subseteq \mathbb{R}$ are intervals, by $\text{length}(I) \cdot \text{length}(J)$. This extends to finite disjoint unions of rectangles: just take

the sum. (Length of I = right endpt − left endpt.)
What about other sets, even just open sets.



We can take the "inner measure" = area of fully inside rectangles and the "outer measure" = the area of rectangles that intersect A and hope that for a fine enough grid these numbers will get arbitrarily close. But for which sets will this happen? The Banach–Tarski ($O = O + O$) paradox in $\mathbb{R}^3$ says that one can't hope that a notion of volume can be defined for all sets, but we at least hope we can do it for open and closed sets.

From analysis. We want to have nice classes of functions that are closed under ctbl operations, in particular limits. Indeed, a pointwise limit of continuous (even smooth) functions may not be Riemann integrable. In particular, we need a better integration theory and larger classes of integrable functions that are closed under ptwise limits.

Measures, their construction, and properties

Polish spaces. A metric space (X, d) is called Polish if it is separable (admits a ctbl dense set) and d is complete (all Cauchy sequences converge).

Examples.

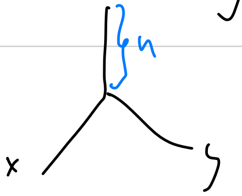
- $\mathbb{R}$ and more generally $\mathbb{R}^d$ with metric $d(\vec{x},\vec{y}) := \|\vec{x}-\vec{y}\|_\infty$, where $\|\vec{x}\|_\infty = \max_{1\le i\le d} |x_i|$. Other metrics of the form $\|\vec{x}-\vec{y}\|_p$, where $1 \le p < \infty$, are equivalent to this metric (produce the same open sets). Here $\|\vec{x}\|_p := \left(\sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}$.

- Closed subsets of Polish spaces are Polish. Indeed, they are still complete with the same metric and separability is hereditary (HW). In particular, $[0,\infty) \subseteq \mathbb{R}$ is Polish.

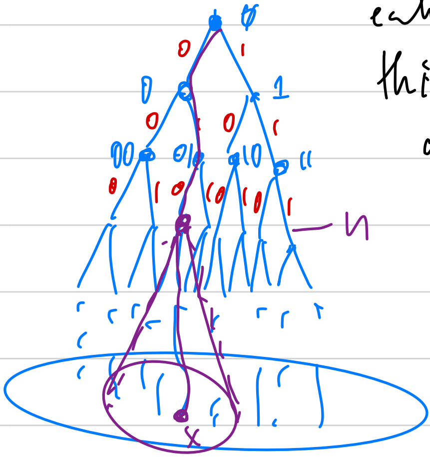
- What about $[0,1)$? No in the usual metric $d(x,y) := |y-x|$ because it's not complete. But we can change the metric to an equivalent one so that is Polish. Take a homeomorphism (a continuous bijection with continuous inverse) between $[0,1)$ and $[0,\infty)$ and copy the metric from $[0,\infty)$ to $[0,1)$.

- In fact, it's a theorem of Descriptive Set Theory that all G_δ subsets of Polish spaces are "Polishable", i.e. admit an equivalent Polish metric. G_δ = ctbl intersection of open sets (e.g. all closed sets in metric spaces are G_δ HW).

The spaces $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$: Let A (alphabet) be a ctbl nonempty set, e.g. $A = \mathbb{N}$, $A = 2 := \{0,1\}$. We define a metric on $A^{\mathbb{N}}$ by $d(x,y) := \frac{1}{2^{n(x,y)}}$, where $n(x,y)$ is the first index $n \in \mathbb{N}$ at which $x(n) \neq y(n)$.



To picture $A^{\mathbb{N}}$, we draw a tree: I'll draw for $2^{\mathbb{N}}$:



each element of $2^{\mathbb{N}}$ is an infinite branch through this tree. The closed ball of radius $\frac{1}{2^n}$ about $x \in A^{\mathbb{N}}$ is

$$\overline{B}_{2^{-n}}(x) := \{ y \in A^{\mathbb{N}} : y|_n = x|_n \}$$

Every element of $\overline{B}_{2^{-n}}(x)$ is a center of this ball.

Balls here are cylinders. For a finite word $w \in A^n$, we define the cylinder at w to be:

$$[w]_A := \{ x \in A^{\mathbb{N}} : x|_n = w \}.$$

We just saw that balls are cylinders. Cylinders are both closed and open. Indeed, cylinders are open balls and a complement of a cylinder is a (ctbl disjoint) union of cylinders. We call them clopen set.

Proposition. $A^{\mathbb{N}}$ is separable.

Proof. Take $Q := \{ wa^{\infty} : w \in A^{<\mathbb{N}} \}$, where $a \in A$ is a fixed letter and a^{∞} is the infinite word $aaaa\ldots$

the set of finite sequence in A

Q is dense because for any cylinder $[w]$, $[w] \cap Q \ni waaa\ldots$ $\square$

Proposition. (a) $A^{\mathbb{N}}$ with this metric is complete, hence Polish.

(b) If A is finite, say 2, $A^{\mathbb{N}}$ is compact.

Proof. HW.

$2^{\mathbb{N}}$ is called Cantor space and $\mathbb{N}^{\mathbb{N}}$ is called Baire space.

Most spaces in analysis are Polish, maybe after switching the metric to an equivalent one. For the rest of the course, we'll work with $\mathbb{R}^d$ and $2^{\mathbb{N}}$ (maybe also $\mathbb{N}^{\mathbb{N}}$).

σ-algebras and measurable spaces.

We like open and closed subsets of metric spaces, but we'd like to also work with their (ctbl) unions, intersections, and their complements.

Def. Let X be a set (think of it as our space, $\mathbb{R}^d$ or $2^{\mathbb{N}}$). An algebra of subsets of X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ that contains $\emptyset$ and is closed under complements and finite unions (hence also finite intersection, by De Morgan's law).
An algebra is called a σ-algebra if it's moreover closed under ctbl unions (hence also ctbl intersections).

Examples.

- Finite unions of cylinders in $2^{\mathbb{N}}$ form an algebra.
- Finite unions of boxes in $\mathbb{R}^d$ form an algebra. By a box we mean a set of the form

$$I_1 \times I_2 \times \dots \times I_d,$$

where each I_j is an interval (a,b) or $(a,b]$, or $[a,b)$ or $[a,b]$. HW

a may be $-\infty$, b may be ∞.